Math 1112

Unit 2: Algebra Basics

2.1 Notation
1. Algebra as a language
2. Beyond numbers: variables, terms, expressions, polynomials

2.2 Arithmetic with algebraic terms – will not be tested directly
3. Distributive property the power of brackets
4. Add and subtract polynomials
5. Multiply and divide monomials
6. Multiply monomials by polynomials
7. Multiply binomials

2.3 Substitution
8. Substitution into mathematical expressions
9. Exercises

2.4 Solve simple equations (one variable)
10. Concepts and procedures
11. Exercises
12. Exercises - continued
13. Translate English equations to algebra (linguistic to symbolic)
14. Percent of as equation

2.5 Beyond simple equations
15. Inequalities and their solution
16. Absolute value

Author:

Introduction to algebra as a language

Algebra, comes from the Arabic ***al - jabr***, which roughly means "the reduction". Through algebra mathematicians map phenomena from the the living and breathing world using numbers and other abstract objects. These numbers an dother symbols are the building blocks of algebra as a language.

Below is a review of a few mathematical objects used in elementary algebra and their English counterparts. The basic building block is the number, but in algebra, numbers can be represented by a letter, indicating that there are a variety of possible numbers possible, or that the particular value is unknown. This building block is called a ***variable***.

Operations: are actions done to numbers and their representatives. Each operation can have more than one English phrase or word to replace it. Below is a list of the basic arithmetic operations. They can be expressed symbolically or in words:

symbol	**Operation (formal name)**	**Operation (informal name)**	**answer**
+	addition	plus, more than, add, increase by	sum
–	subtraction	subtract, difference, take away, decrease by	difference
× or · or *	multiplication	times, multiply	product
/ or ÷	division	divided by (over)	quotient

Translating from English to algebra:

Read the table below carefully. Notice how the comma in the English phrase changes the mathematical expression

Note: 5r is the same as 5 × r, 7k is the same as 7 × k, wk is the same as w × k, or w•k, or w*k

English Phrase	**symbolic/algebraic Expression**
three more than six;	$3 + 6$
five times a number plus three;	$5 \times r + 3$ or $5r + 3$
five times, a number plus three;	$5 \times (r +3)$ or $5(r + 3)$
two divided by a number;	$2 \div w$
a number divided by 3 added to 7;	$y \div 3 + 7$
twice a number take away twelve;	$2q – 12$

Exercise 1: Translate the following into mathematical expressions:

a) The sum of 7 and a number

b) 4 minus a number

c) 9 divided by a number

d) 3 less than a number

e) 7 more than 6 times a number

f) twice a number plus 5

g) a number multiplied by 3

h) a number subtracted from 4

i) a number divided by 9

j) the product of 5 and a number, increased by 8

k) two more than three times a number

l) if n is a whole number what is the next whole number?

Beyond Numbers: variables, terms, expressions, polynomials

An algebraic ***term*** is made up of a number (numerical coefficient) and/or a 'letter' (called a ***variable***/literal coefficient)

Examples: 3x   4y   y   12 $17w^2$

An algebraic *expression* is made up of a one or more algebraic terms linked by + or – .

Types of expressions:

Monomial – one term	3x
Binomial – two terms	3x – 2q
Trinomial – three terms	7 – q + 4z
Polynomial – many terms	all the above and more

Exercise 1: Identify the following as a monomial binomial or trinomial:

a. $4x^2$ b. $3x - y$ c. $24xy^2z$ d. $3a + 4b + 5c$ e. $3 + a$

Adding and subtracting like terms: You can only add and subtract like terms. ***Like terms*** are those that have the same literal coefficients.

3x & 4x	are like	so $3x + 4x = 7x$
3x & 4y	are not like	so $3x + 4y = 3x + 4y$ (no change)
3x & $4x^2$	are not like	so $3x + 4x^2 = 3x + 4x^2$ (no change)

note: expressions should be in alphabetical order with highest exponents first.

Exercise 2: simplify the expressions by adding and subtracting like terms.

a. $3x + 4x^2 + 5x$ b. $3x + 4y + x$ c. $3x + 4x + 4y$ d. $x^2 + x + y^2 + 7x$

e. $7x^2y - 4x^2 - 5xy^2$ f. $12x^2 - 12y^2$ g. $21xy^2 - 21xy^2$ h. $5x^2yz + 4xyz - x^2yz$

i. $3x - 4x^2 - 5x$ j. $3x - 4y - x$ k. $3x - 4x - 4y$ l. $x^2 - x - y^2 - 7x$

The power of brackets (Brackets help organize the order in which we conduct operations.) Completing the operations inside brackets makes much less work where multiple steps are involved.

Examples: Look at the following as evaluated; pay attention to the order:

a) $23 - 2(4+5)$
$= 23 - 2(9)$
$= 23 - 18$
$= 5$

b) $(23 - 2)\times(4+5)$
$= 21 \times 9$
$= 189$

c) $(4^2 + 75) / 5^2$
$= (16 + 75) / 25$
$= 91 \div 25$
$= 3.64$

d) $4^2 + 75 / 5^2$
$= 16 + 75 / 25$
$= 16 + 3$
$= 19$

Exercise 1: Evaluate the following: (do the work in your notebook)

a) 3×10^5 b) $(7 - 2)^2$ c) $7^2 - 2^2$ d) $36 - 10 \times 3$

e) $19 - 5(7 - 4)$ f) $22 \times 3 - 5$ g) $3(22 - 4)$ h) $47 - 2(4 + 6)$

i) $67 - (3 \times 4 - 5) \times 6$ j) $7 + 0.05 \times 10^2$ k) 3.1015×10^4

Add and subtract polynomials:

It gets a bit more complicated when you see non-number terms with brackets around them, and when multiple operations are involved.

When adding or subtracting terms in brackets: i.e. there is a + or – sign in front of ( ), then think of it as adding or subtracting each term inside (see distributive property on next page).

Example 1: a) $3x + 4 - (5x - 3)$

$= 3x + 4 - 5x + 3$

$= 3x - 5x + 4 + 3$

$= -2x + 7$

b) $4x - (3x + 2 - 7y^2)$

$= 4x - 3x - 2 + 7y^2$

$= x + 7y^2 - 2$

<u>Exercise 1:</u> simplify the following expressions

a. $26x + 38 - (9x + 24)$

b. $38x^2 + 9x + (45x + 6x^2)$

c. $6x^2 + 7x - (22x + 47y)$

d. $(43 + 20x) + (25 + 38x^2)$

e. $42 + 10x - (28x^2 - 44x)$

f. $(11x^2 + 14x) - (-39x + 16 - 18x^2)$

g. $39x + 20 - (26x^2 + 31x)$

h. $43 + 35x^2 - (13xy + 21)$

i. $12x + 21 + (34y - 8x + 43)$

j. $6x + 20x^2 + (-3x + 26x^2)$

k. $14 + 45x^2 + (-16x + 20x^2)$

Multiply and divide monomials

You may wish to review working with powers and exponents before starting on this module.

Multiplication short cut: when multiplying powers with the same base, add the exponents.

Example 1

a. $3x^2 \times 4x^3 = (3 \times 4)x^{2+3}$
$= 12x^5$

b. $7q^3 \times 3q^{-2} = (7 \times 3)x^{3-2}$
$= 21x^1$
$= 21x$

Division short cut: when dividing powers with the same base subtract the exponents.

Example 2: simplify the following expression

$12x^2 / 2x^4 = (12 / 2)x^{2-4}$
$= 6x^{-2}$
$= \frac{6}{x^2}$ {this last step is not necessary, but it illustrates an equivalent of $6x^{-2}$)

<u>Exercise 1:</u> simplify the following expressions

a) $x^7 \times x^3$

b) x^7 / x^3

c) $5x^7 \times 4x^3$

d) $12x^7 / (-3) x^3$

e) $5ac^2d \times 12a^2c^{-1}$

f) $9yz^3 / 4xz^{-1}$

g) $2ab \times 7a^7b^{-2}$

h) $75x^4y^5 / 3x^4y^5$

Multiply monomials by polynomials the distributive property

The key is to use the multiplication rules for exponents you reviewed in the previous page and make sure that you multiply the monomial by each term of the polynomial. Beware of the negative sign.

Multiplication and division with monomial and binomial

Example 1. Make sure that you multiply each 'term' of the binomial by the monomial.

a. $3(4x - 5) = 3\,(4x) + 3(-5)$
$= 12x - 15$ (stop here - the terms '$12x$' and '-15' are not like)

The above is an example of the distributive property (i.e. by following the arrows you distribute the three to every term inside the bracket). The above is also a simple example of multiplying a monomial (one term expression) by a binomial (two term expression).

Example 1b below is a bit more complicated as in it the monomial has a literal coefficient (x).

b. $3x(4x - 5) = 3x(4x) + 3x\,(-5)$
$= 12x^2 - 15x$ (stop here the terms '$12x^2$' and '$-15x$' are not like)

<u>Exercise 1:</u> simplify the following expressions

a) $3(\,3x + y - z)$

b) $4y(3x^2y + 7y)$

c) $7x^{21}y^{44}(x - y)$

d) $3a^2 + 3(2a^2 - 14a)$

e) $2a^2b^3 + 4a(3a + 7ab^3)$

f) $7x^4(4z - y + 3x)$

Multiply binomials

When multiplying binomials we need to keep in mind the distributive property and make sure that we multiply each term in one binomial by each term in the other. Mnemonics like FOIL (for First, Outside, Inside and Last) help us keep track to make sure we don't miss anything, but you could just as easily use LOIF, or OLIF. If you don't believe me try each to see if you get the same answers

Example 1: simplify the following expression (first using FOIL, then OLIF)

FOIL

$(3 + x)(a - b) = 3(a) + 3(-b) + x(a) + x(-b)$
$= 3a - 3b + ax - bx$
$= 3a - 3b + ax - bx$

OLIF

$(3 + x)(a - b) = 3(-b) + x(-b) + x(a) + 3(a)$
$= -3b - bx + ax + 3a$
$= 3a - 3b + ax - bx$

<u>Exercise 1:</u> simplify the following expressions

a) $(x + 1)(x - 3)$

b) $(x + 5)(x - 2)$

c) $(-3x + 4)(-3x + 4)$

d) $(4x - 1)(4x + 1)$

e) $(3a + b)(a - b)$

f) $(7 + 2)(x - 2)$

g) $(x + 1)(x - 1)$

h) $(x + 1)(x + 1)$

Substitution into expressions: in mathematics when you know the value of the variable you can evaluate the value for of the whole expression.

		Notes
Example 1: Evaluate the expression $5 + a$ when $a = 2$		
Solution: when $a = 2$,	$5 + a = 5 + 2$	*what you are doing here is*
	$= 7$	*replacing the "a" with a "2"*
Example 2: Evaluate the expression $4q$ when $q = –7$.		
Solution: when $q = –7$	$4q = 4 \times –7$	*remember that 4q means*
	$= –28$	*"4 times q"*
Example 3: Evaluate $3(a–b)$ when $a = 9$ and $b = 2$.		*Distributive property is important*
Solution: when $a = 9$ & $b = 2$,	$3(a – b) = 3(9 – 2)$	*to get the right answer.*
	$= 3(7)$	
	$= 21$	
Example 4: Evaluate $3d + d^2$ when $d = 5$		
Solution: when $d = 5$	$3d + d^2 = 3 \times 5 + 5^2$	*slow thinking*
	$= 3 \times 5 + 25$	
	$= 15 + 25$	
	$= 40$	

Now get ready for a lot of practice:

Exercises 1: Find the value of y in the following

a) $y = x + 2$ when $x = 4$	b) $y = 6 \div c$ when $c = –3$	c) $y = 4g$ when $g = 10$
d) $y = 8f$ when $f = 30$	e) $y = 3b + 2$ when $b = 1$	f) $y = 6d – 7$ when $d = –3$
g) $y = 3p – 4q$; $p = 3, q = 2$	h) $y = 6f + 7g$; $f = 7, g = 3$	i) $y = 2x^3 – 4x$ when $x = 4$
j) $y = 5(w – 2) +2w$; $w = –10$	k) $y = p + p^2 + p^3$; $p = 3$	l) $y = 6t + 4t^2 – 2(t +3)$; $t = 5$

Substitution - exercises

Exercise 2: If $a = 2$, $b = -1$, $c = 3$, $d = 4$, evaluate that following expressions.

a) $a + 4$ b) $d \div a$ c) $d - c + a$ d) $2b + 4c$

e) $6d - 5a$ f) d^2 g) c^3 h) $cd - a^2$

Exercises 3 to 4 are a bit more difficult for those that don't know the operations described – but if you read the instructions carefully and think slowly you will succeed.

Exercise 3: a! is an operation called factorial and is defined as follows for whole numbers (i.e. no negatives).

$a! = a(a-1)(a-2)\ldots.(2)(1)$ **For example:** $4! = 4\times3\times2\times1= 24$; $0! = 1$.

Calculate the following for $a = 7$

a) $a!$ b) $(a-1)!$ c) $(a+2)!$ d) $(a-5)!$

Exercise 4: $\binom{n}{x}$ is an operation called 'n choose x' where $n \geq x$. $\binom{n}{x} = \frac{n!}{(n-x)!x!}$

Calculate the following:

a) $\binom{7}{2}$ b) $\binom{5}{3}$ c) $\binom{7}{1}$ d) $\binom{9}{4}$

Exercise 5: $\sum_{i=1}^{6} i$ means the sum of all i starting from i = 1 all the way up to 6

That is: $\sum_{i=1}^{6} i = 1 + 2 + 3 + 4 + 5 + 6 = 21.$

Calculate the following:

a) $\sum_{i=2}^{7} i =$ b) $\sum_{i=1}^{6} i^2 =$

c) $\sum_{i=1}^{6}(2i) =$ d) $\sum_{i=1}^{6}(i + 2)$

Equations: An equation is a statement of equality between one algebraic expression and another.

Is each of the mathematical expressions in the left column an equation?

	yes	no
$3x = y$		
$2x - 5$		
$3 - 2 = q$		
$45x^2 - 7x + 4 = 0$		
$5 = 7$		

Solving equations with mental math (recognition).

To solve an equation with one variable requires us to find its value. At times you will be able to solve equations by just looking at the question (or trial and error)

e.g. $3 + x = 7$; You can see that $x = 4$. This is true since $3 + 4 = 7$;

Solving equations can be seen as conducting a balancing act. The only Rule you need: Whatever you do to one side of the equation you must do to the other.

We will only be solving equations with one variable i.e. only 'x' or only 'y' or only 'q' etc.

Step 1: Simplify each side first. i.e. gather up all variables, clear out the brackets.
Step 2: Work at 'deuglifying' the side that bugs you the most using basic operations.
Step 3: Whatever you did to one side you must do to the other
Step 4: Keep doing step 2 and step 3 until you have isolated the variable.

Hint: If you are uncomfortable with fractions just change them to decimals. Note: beware of repeating decimals. $1/3 = 0.33\dot{3}$,

Example1: Every line is a new step. I have shaded the balancing operations.

a. $\frac{3x}{3} = \frac{6}{3}$ {simply divide both sids by 3} $x = 2$ {you are done }	b. $3x + 7 = 6$ { get rid of 7 first} $3x + 7 - 7 = 6 - 7$ $\frac{3x}{3} = \frac{-1}{3}$ $x = -0.33\dot{3}$ or $x = -\frac{1}{3}$
c. $\frac{x}{4} = 5$ {multiply both sides by 4} $4 \cdot \frac{x}{4} = 5 \cdot 4$ $x = 20$ {done}	d. $\frac{1}{4}x = 5$ {same as c) but this time will convert to decimal} $\frac{0.25x}{0.25} = \frac{5}{0.25}$ {now divide both sides by 0.25} $x = 20$ {finito}

Solve Equations - Practice

Exercise 1: Solve the following equations:

a) $m + 4 = 5$ b) $s - 7 = 9$ c) $y - 7 = -16$

d) $5a = 60$ e) $5a = -60$ f) $d + 17 = 40$

g) $\frac{m}{5} = 20$ h) $n/3 = 18$ i) $\frac{20}{m} = 5$

j) $2w + 5 = 17$ k) $2r - 7 = -1$ l) $15 - 3k = 0$

m) $15 + k = 16 - k$ n) $15 + r = 14$ o) $q - 5 = -12$

Solve Equations - Exercise 1: continued

p) $3m - 1 = 5$

q) $3s - 7 = -16$

r) $7 - 2k = -1$

s) $7a + 3 = 63 + 2a$

t) $9m = 121 - 2m$

u) $2(m/4 + 1) = 42$

v) $\frac{16-3k}{4} = 1$

w) $y - 1/3 = 1/3$

x) $4(3r - 7) + 8 = -8$

y) $\frac{m}{12} = \frac{6}{9}$

z) $\frac{4}{7} = \frac{h}{22}$

aa) $\frac{22}{7} = \frac{51}{x}$

Translate equations from English into algebra (linguistic to symbolic)

The problems you will be solving here will involve translating English phrases into algebraic phrases and equations, and solving for the unknown. Remember also to use your skills from page 1.

Example 1: When a certain *number* is doubled and 10 is added, the result is 28
Find the number.

solution - step 1 let the *number* be x,
step 2 translate the words into algebra to get $2x + 10 = 28$
step 3 solve for x, to get $x = 9$.

Exercise 1: Express the following in algebraic/symbolic form and solve the problem.

a. A certain number increased by 32 is 96. Find the number.

b. Fifteen less than four times a number is 33. Find the number.

c. The sum of two consecutive numbers is 81. What are they?

d. When 9 is added to $\frac{1}{4}$ of a number, the result is 20. Find it.

e. Four times a number is 72. What is the unknown number?

f. There are 3 unknown numbers J is 2 greater than C, which is 5 greater than A. The sum of all 3 is 42. Find each number.

g. John is thinking of a number. If he multiplied it by 6 and subtracts 4, the result is 90. Find the number.

The basic percent equation:

All problems with percentage can be boiled down to the basic percent equation. If you aren't sure what a particular question is asking make sure to take a breath and think slow, then use the guide below to help you set it up.

$$\square\ \%\ \text{of}\ \square = \square$$

There are 3 types of percent problems, all can be solved using the basic percent equation above.

Example 1: Find 27% of 35

27% of 35 = ?? { use basic % equation}

$0.27 \times 35 = x$ {convert % to decimal}

$9.45 = x$

Example 2: 105% of a number is 32. Find the number.

105% of ?? = 32 { basic % equation}

$1.05 \times x = 32$ {convert % to decimal}

$\frac{1.05x}{1.05} = \frac{32}{1.05}$ ***{divide both sides by 1.05}***

$x = 30.4762$

$x \approx 30.48$

Example 3: What % of 55 is 42?

??% of 55 = 42? ***{here the % is unknown}***

$x\ \% \times 55 = 42$

$\frac{55x}{55}\ \% = \frac{42}{55}$ {divide both sides by 0.55}

$x = 0.763636$ { convert to % }

$x \approx 76.36\%$

Exercise 1: Solve for Q in each of the following:

a) 34% of 27 is Q	b) 27% of Q is 34	c) Q% of 176 is 45
d) 145% of 145 is Q	e) 50% of Q is 46	f) Q% of 44 is 22
g) Q is 24% of 77.99	h) 24.32 is 15% of Q	i) 26 out of 44 is Q%

Inequalities and their solution

Reminder of sign notation: $>$ greater than; $<$ less than;

$\geq$ greater than or equal; $\leq$ less than or equal

Example 1: write out all integers that fit the following: ≥ -1 and < 8

Solution: -1, 0, 1, 2, 3, 4, 5, 6, 7

<u>Exercise 1:</u> For each of the following write out all integers that fit the criteria:

a) >5 and < 7 b) <6 and > -3 c) ≤ 6 and >5

Solving Inequalities:

Basically solving inequalities works just like solving equations except for 2 details.

Detail 1. when multiplying (or dividing) both sides of an inequality by a negative number change the direction of the inequality.

$3x < 5$ *action: multiply both sides by negative 2*

$-2*3x < 5 * -2$

$-6x > -10$

Detail 2. when flipping the inequality ↔ you need to change the direction of the inequality.

$3x < 5$ flip ↔

$5 > 3x$

Example 1: Solve for x in the following inequalities.

a) $3x + 5 < 7$

$3x + 5 - 5 < 7 - 5$

$\frac{3x}{3} < \frac{2}{3}$

$x < 2/3$

b) $-7x > 2x - 5(x + 2)$

$-7x > 2x - 5x - 10$

$7x > -3x - 10$

$-7x + 3x > -3x + 3x - 10$

$-4x > -10$

$\frac{-4x}{-4} < \frac{-10}{-4}$

$x < 2.5$

remember that the sign switches direction when dividing by a negative (in this case -4)

<u>Exercise 2:</u> Solve for x in the following.

a) $9x + 2 < 7$ b) $4x + 4 >4$ c) $3x - 2 < -3(2x + 5)$ d) $(16 - 3k)/4 >1$

Absolute value: $|x|$ absolute value (makes the number positive).

The only tricky thing about reading absolute value is that you need to figure out what the value of the inside is before applying the function to its contents. Treat it like a bracket – solve what is in the bracket first – then apply the 'absolute'. See example c)

Example 1: a) $|4| = 4$ b) $|-4| = 4$ c) $|-7-4| = |-11|$
$= 11$

Exercise 1: Evaluate the following:

a) $|-77|$ b) $|3 - -4|$ c) $|3 - 4|$ d) $|5 - 47384|$

e) $|3x - 4x^2 + 5x^3|$ where $x = -1$ f) $|-3 - {}^{-}4|$

Convert one rate to another (technique): There is a particular type of problem that comes up in many quantitative problems. If you recognize it and set up a *statement of equality* between two ratios, known as a proportion.

e.g. $\frac{m}{5} = \frac{7}{12}$; you have seen a few of these in the solve equations exercises.

The short cut technique to solving for the unknown in this case is to cross multiply

Example 1: $\frac{m}{5} \times \frac{7}{12}$

multiply to get $12m = 35$ *next: divide both sides by 12*

now solve for m $m = \frac{35}{12}$

$m = 2.916666$

Exercise 1: Solve for the unknown in the following:

a) $\frac{m}{5} = \frac{3}{15}$ b) $\frac{5}{m} = \frac{7}{12}$ c) $\frac{3x}{2} = \frac{1}{9}$

d) $\frac{3}{2x} = \frac{5}{2}$ e) $\frac{2}{9} = \frac{7}{x}$ f) $\frac{200}{5455} = \frac{x}{10}$